Int. J. Solids Structures Vol. 26, No. 4, pp. 375–390, 1990
Printed in Great Britain.

0020–7683/90 $3.00+ .00

ELLIPTIC INTEGRAL SOLUTIONS OF PLANE ELASTICA WITH AXIAL AND SHEAR DEFORMATIONS

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(*Received* 9 *November* 1988; *in revised form* 1 *May* 1989)

Abstract—Closed-form solutions are derived for elastica with axial and shear deformations, using elliptic integrals. The theories used here are the Timoshenko beam theory of finite displacements with finite strains and that with small strains. Elliptic integral solutions are further transformed to normal forms to obtain accurate solutions. As a result, both of the closed-form solutions are expressed in normal forms from the first to the third kind of elliptic integrals. With these solutions, two kinds of structures are analyzed to examine the effect of shear deformations along with the accuracy of the approximate theory of finite displacements with small strains.

1. INTRODUCTION

The exact governing equations for the finite displacement beam theory become highly nonlinear and, hence, it is very difficult to solve these equations analytically. Thus, for practical purposes, problems of this kind are usually solved by the methods based on the finite element approximations.

However, the closed-form solutions for these problems are still important to the practical point that the accuracy of the approximate methods can be precisely evaluated by these solutions, to say nothing of the mathematical importance. It is well known that the closed-form solutions can be obtained for elastica problems, using elliptic integrals. However, the elliptic integral solutions presented so far were primarily for the inextensional elastica, where the elongation of a centroidal axis is ignored: see Timoshenko and Gere (1961), Sliter and Boresi (1964), Lee *et al.* (1968), Britvec (1973), Law (1982) and Seide (1984), among others. Different from the above solutions ignoring axial deformation, we recently derived the general closed-form solutions of plane elastica, precisely considering the deformation of centroidal axis (Goto *et al.*, 1987). In this derivation special efforts were made to reduce the elliptic integrals to normal forms, i.e. normal forms from the first to the third kind. As a result, highly accurate solutions were successfully obtained for extensional elastica.

Herein, we further derive the closed-form solutions of elastica, taking into account shear deformation in addition to axial deformation. In this derivation, we utilize the theory presented by Reissner (1972) which exactly considers the geometrical nonlinearity within the framework of the Timoshenko beam theory. In view of the practical importance, we also derived the solutions under the assumption of small strains, since strains of beams are usually small compared with unity even when beams undergo large displacements.

The closed-form solutions, derived here, are expressed in terms of elliptic integrals. Thus, following the same procedure as used by Goto *et al.* (1987), they are transformed to normal forms to obtain accurate solutions.

With the closed-form solutions so obtained, we analyzed a few simple structures in order to demonstrate the accuracy of the present solutions as well as to examine the effect of shear deformation. These numerical results are also intended to be used as bench marks to validate the accuracy of approximate numerical analyses.

To integrate the governing differential equations in closed-form, the cross-section and the initial curvature of beams are assumed constant. Further, it is assumed that beams are only subjected to concentrated loads.

2. GOVERNING EQUATIONS OF THE TIMOSHENKO BEAM UNDER FINITE DISPLACEMENTS

Reissner (1972) was the first to present a finite displacement theory of the curved Timoshenko beam, where the geometrical nonlinearity is precisely taken into account under the assumption of the Timoshenko beam. Later, Iwakuma and Kuranishi (1984) and Chaisomphob *et al.* (1986) derived the same theory from continuum mechanics, with the help of variational calculus.

Reissner presented in his theory a constitutive model and showed a procedure to determine the elastic constants through experiments. Indeed Reissner's procedure is accurate, but it is cumbersome to carry out an experiment to obtain constitutive equations. Thus, for simplicity we adopt here the theoretical approach used by Chaisomphob *et al.* (1986) to obtain the constitutive relation, based on Reissner's model.

In order to express the governing equations, two kinds of coordinates shown in Fig. 1 are introduced here. One is Cartesian coordinates (y, z) fixed in space and the other is the orthogonal curvilinear coordinates (n, s) with s taken along the centroidal axis of a curved member *before* deformation.

Using these coordinates, the governing equations of the Timoshenko beam are shown in Table 1, classified according to the theories. The theory of (a) Finite Displacements with Finite Strains corresponds to Reissner's theory with the constitutive equations of Chaisomphob *et al.* (1986), while the theory of (b) Finite Displacements with Small Strains is derived from the theory of (a), approximated by the condition of small strains.

Here, we briefly explain the above theories along with the notations.

The kinematic relations for the above theories are obtained, using the assumption of the Timoshenko beams. This assumption is that the transverse plane originally normal to the centroidal axis remains plane *after* deformation. However, it should be noted in the Timoshenko beam theory that the transverse plane does not necessarily remain normal to the centroidal axis, different from the Bernoulli–Euler beam theory. Thus, *after* deformation, the angle between the tangent of the centroidal axis and the transverse plane originally normal to the centroidal axis is expressed by $\pi/2$-Λ, as shown in Fig. 1. The angle Λ denotes the shear deformation of the beam. Henceforth, the transverse plane originally normal to the centroidal axis is referred to as the cross-section.

If we define α_0 and α as the angles between z-axis and the tangent of the centroidal axis, respectively, *before* and *after* deformation, these angles can be related to the displacements on the centroidal axis as

$$v_0' = \sqrt{g_0}\sin\alpha - \sin\alpha_0,$$
$$w_0' = \sqrt{g_0}\cos\alpha - \cos\alpha_0, \qquad (1a,b)$$

in which v_0 and w_0 are the y- and the z-components of the translational displacements on the centroidal axis and g_0 is given by

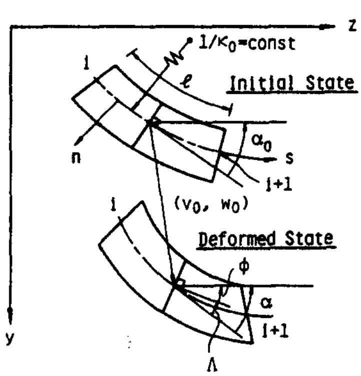

Fig. 1. Geometry of the initial and the deformed beam element.

Table 1. Timoshenko beam theories

Theories	Equilibrium equations	Constitutive relations	Boundary conditions: Mechanical	Boundary conditions: Geometrical
(a) Finite Displacements with Finite Strains	$\{N\cos(\alpha-\Lambda)-V\sin(\alpha-\Lambda)\}'=0$	$N=E\bar{A}(\sqrt{g_0}\cos\Lambda-1)$	$N\cos(\alpha-\Lambda)-V\sin(\alpha-\Lambda)=F_{z\beta}$	$v_0=v_{0\beta}$
	$\{N\sin(\alpha-\Lambda)+V\cos(\alpha-\Lambda)\}'=0$	$V=kG\bar{A}\sqrt{g_0}\sin\Lambda$		
	$M'-\sqrt{g_0}(V\cos\Lambda-N\sin\Lambda)=0$	$M=-E\bar{I}(\alpha'-\Lambda'+\kappa_0)$	$N\sin(\alpha-\Lambda)+V\cos(\alpha-\Lambda)=F_{y\beta}$	$w_0=w_{0\beta}$
(b) Finite Displacements with Small Strains	$\{N\cos(\alpha-\Lambda)-V\sin(\alpha-\Lambda)\}'=0$	$N=E\bar{A}(\sqrt{g_0}-1)$	$M=M_\beta$	$\alpha-\Lambda=\phi_\beta$
	$\{N\cos(\alpha-\Lambda)+V\sin(\alpha-\Lambda)\}'=0$	$V=kG\bar{A}\Lambda$	$(\beta=i,\ i+1)$	$(\beta=i,\ i+1)$
	$M'-(V-N\Lambda)=0$	$M=-E\bar{I}(\alpha'-\Lambda'+\kappa_0)$		

Remarks: the following notations are used throughout tables:

$$\bar{A}=\int_A\frac{1}{1+\kappa_0 n}\,\mathrm{d}A,\quad \bar{I}=\int_A\frac{n^2}{1+\kappa_0 n}\,\mathrm{d}A,$$ (F_{zi}, F_{yi}, M_i) = Mechanical boundary values at node i, $(\bar{v}_{0i}, \bar{w}_{0i}, \phi_i)$ = Geometrical boundary values at node i.

$$g_0 = (v'_0+\sin\alpha_0)^2+(w'_0+\cos\alpha_0)^2. \tag{2}$$

In eqns (1) and (2), prime $'$ denotes the differentiation with respect to s and this notation is used hereinafter. It should be noted that α'_0 is constant, since the initial curvature κ_0 of the centroidal axis given below is constant from the assumption:

$$\alpha'_0 = -\kappa_0. \tag{3}$$

Equilibrium equations in Table 1 are derived based on the above kinematic field. The first two are the equations of the force equilibrium decomposed into the y- and the z-directions. The third is the equation of the moment equilibrium, which newly appears in the Timoshenko beam theory. Unlike the Bernoulli–Euler beam, the moment equilibrium equation becomes independent due to the new degree of freedom added to express shear deformation.

Constitutive equations adopted here are those derived theoretically by integrating the assumed stress–strain relationship over the cross-sectional area. The strain components used here are given by

$$\begin{aligned} e &= \{\sqrt{g_0}\cos(\Lambda)-1-n(\alpha'-\Lambda'+\kappa_0)\}/(1+\kappa_0 n), \\ \gamma &= \sqrt{g_0}\sin(\Lambda)/\{2(1+\kappa_0 n)\}, \end{aligned} \tag{4a, b}$$

where e is the axial strain defined as the component of extensional rate normal to the transverse cross-section, while γ is the shear strain which corresponds to the physical component defined by Fung (1965). On the centroidal axis, strain components of eqns (4) exactly coincide with those shown by Reissner. With these strain components, the simplest constitutive relations are given as follows, using Young's modulus E and shear modulus G:

$$\sigma = Ee, \quad \tau = 2G\gamma, \tag{5a, b}$$

in which σ and τ are the normal and the shear stresses, respectively, defined in terms of the deformed cross-section originally normal to the centroidal axis.

Consequently, the constitutive equations for the theory of (a) Finite Displacements with Finite Strains are obtained by substituting eqns (4) and (5) into the following definitions of sectional forces and moment:

$$N = \int_A \sigma\,\mathrm{d}A, \qquad M = \int_A \sigma n\,\mathrm{d}A, \qquad V = \int_A \tau\,\mathrm{d}A, \tag{6a–c}$$

where $\int_A(\cdot)\,\mathrm{d}A$ denotes the integration over the cross-sectional area. It should be noted, however, in the constitutive relation between the shear force and the shear strain that the shear coefficient k is introduced to improve the accuracy, since the distribution of the shear stress obtained from eqns (4b) and (5b) is considerably different from the actual distribution.

To simplify the constitutive relations, the location of the centroidal axis, i.e. the origin of coordinate n, is selected such that

$$\int_A \frac{n}{1+\kappa_0 n}\,\mathrm{d}A = 0. \tag{7}$$

The theory of (b) Finite Displacements with Small Strains is derived from the above theory of (a), employing the conditions of small strains expressed as

$$\sqrt{g_0}-1 \ll 1, \qquad \Lambda \ll 1. \tag{8a,b}$$

The above conditions are used to simplify the moment equilibrium equations as well as the constitutive relations. Constitutive relations can be obtained by introducing the following strain components simplified by the use of eqns (8):

$$e = \{\sqrt{g_0}-1-n(\alpha'-\Lambda'+\kappa_0)\}/(1+\kappa_0 n),$$
$$\gamma = \Lambda/\{2(1+\kappa_0 n)\}. \tag{9a,b}$$

3. INTEGRATION OF THE GOVERNING EQUATIONS

For simplicity, the integration procedure is shown primarily for the theory of (a) Finite Displacements with Finite Strains, since this procedure is valid for the other theory.

Integrating the force equilibrium-equations and introducing the mechanical boundary conditions at node i lead to

$$N\cos\phi - V\sin\phi = F_{zi},$$
$$N\sin\phi + V\cos\phi = F_{yi}, \tag{10a,b}$$

where the new variable ϕ is used to express $(\alpha-\Lambda)$. ϕ is interpreted as the angle between the deformed transverse plane and the z-axis. Equations (10) can be solved for N and V as

$$N = F_{zi}\cos\phi + F_{yi}\sin\phi,$$
$$V = F_{yi}\cos\phi - F_{zi}\sin\phi. \tag{11a,b}$$

The following relations are given by the constitutive equations in Table 1.

$$\sqrt{g_0}\cos\Lambda = N/E\tilde{A}+1, \quad \sqrt{g_0}\sin\Lambda = V/kG\tilde{A}, \quad \phi' = -M/E\tilde{I}-\kappa_0. \tag{12a–c}$$

Substitution of eqns (12) into the equation of moment equilibrium in Table 1 yields

$$M' - V(N/E\tilde{A}+1) + NV/kG\tilde{A} = 0. \tag{13}$$

Equation (13) can be rewritten as follows, making use of eqns (11) and (12):

$$-EI\phi'' - (F_{yi}\cos\phi - F_{zi}\sin\phi)$$
$$- (F_{yi}\cos\phi - F_{zi}\sin\phi)(F_{zi}\cos\phi + F_{yi}\sin\phi)\left(\frac{1}{EA} - \frac{1}{kGA}\right) = 0. \tag{14}$$

Equation (14) is integrated by multiplying both sides by ϕ'. Introducing the boundary conditions at node i, the integrated equation is given in the form

$$(\phi')^2 = \left(\frac{M_i}{E\tilde{I}} + \kappa_0\right)^2 - \frac{2F_{yi}}{E\tilde{I}}(\sin\phi - \sin\phi_i) - \frac{2F_{zi}}{E\tilde{I}}(\cos\phi - \cos\phi_i)$$
$$- \frac{1}{E\tilde{I}}\left(\frac{1}{E\tilde{A}} - \frac{1}{kG\tilde{A}}\right)\left\{F_{yi}F_{zi}(\sin 2\phi - \sin 2\phi_i) + \frac{F_{zi}^2 - F_{yi}^2}{2}(\cos 2\phi - \cos 2\phi_i)\right\}. \tag{15}$$

This equation is solved for ϕ' as

$$\phi' = f, \tag{16a}$$

$$f = -\mathrm{sign}\,(M/E\tilde{I}+\kappa_0)\left[\left(\frac{M_i}{E\tilde{I}}+\kappa_0\right)^2 - \frac{2F_{yi}}{E\tilde{I}}(\sin\phi-\sin\phi_i) - \frac{2F_{zi}}{E\tilde{I}}(\cos\phi-\cos\phi_i) - \frac{1}{E\tilde{I}}\left(\frac{1}{E\tilde{A}}-\frac{1}{kG\tilde{A}}\right)\left\{F_{yi}F_{zi}(\sin 2\phi-\sin 2\phi_i) + \frac{F_{zi}^2-F_{yi}^2}{2}(\cos 2\phi-\cos 2\phi_i)\right\}\right]^{1/2}. \tag{16b}$$

where sign $(\cdot)$ takes ± 1 according to the $\pm$ of $(\cdot)$.

Equations (16) can further be integrated as

$$s-s_i = \int_{\phi_i}^{\phi} (1/f)\,\mathrm{d}\phi. \tag{17}$$

Equation (17) is used to calculate the rotational angle of the cross-section.

Next derived are the equations to calculate displacements. Equations (1) are rewritten as

$$(v_0+y_0)' = \sqrt{g_0}\,(\cos\Lambda\sin\phi+\sin\Lambda\cos\phi),$$
$$(w_0+z_0)' = \sqrt{g_0}\,(\cos\Lambda\cos\phi-\sin\Lambda\sin\phi), \tag{18a,b}$$

where (y_0, z_0) is the coordinates of the centroidal axis *before* deformation.

After the substitution of eqns (11), (12) and (16), eqns (18) can be integrated in the form

$$v_0+y_0 = v_{0i}+y_{0i}+F_{zi}\left(\frac{1}{E\tilde{A}}-\frac{1}{kG\tilde{A}}\right)\int_{\phi_i}^{\phi}\frac{\sin\phi\cos\phi}{f}\,\mathrm{d}\phi + F_{yi}\left(\frac{1}{E\tilde{A}}-\frac{1}{kG\tilde{A}}\right)\int_{\phi_i}^{\phi}\frac{\sin^2\phi}{f}\,\mathrm{d}\phi + \int_{\phi_i}^{\phi}\frac{\sin\phi}{f}\,\mathrm{d}\phi + \frac{F_{yi}}{kG\tilde{A}}(s-s_i), \tag{19a}$$

$$w_0+z_0 = w_{0i}+z_{0i}+F_{yi}\left(\frac{1}{E\tilde{A}}-\frac{1}{kG\tilde{A}}\right)\int_{\phi_i}^{\phi}\frac{\sin\phi\cos\phi}{f}\,\mathrm{d}\phi - F_{zi}\left(\frac{1}{E\tilde{A}}-\frac{1}{kG\tilde{A}}\right)\int_{\phi_i}^{\phi}\frac{\sin^2\phi}{f}\,\mathrm{d}\phi + \int_{\phi_i}^{\phi}\frac{\cos\phi}{f}\,\mathrm{d}\phi + \frac{F_{zi}}{E\tilde{A}}(s-s_i). \tag{19b}$$

Then, we proceed to derive the equation to calculate the sectional moment. Noting eqns (11) and (16), eqn (13) can be easily integrated as

$$M = M_i + F_{yi}\int_{\phi_i}^{\phi}\frac{\cos\phi}{f}\,\mathrm{d}\phi - F_{zi}\int_{\phi_i}^{\phi}\frac{\sin\phi}{f}\,\mathrm{d}\phi + (F_{yi}^2-F_{zi}^2)\left(\frac{1}{E\tilde{A}}-\frac{1}{kG\tilde{A}}\right)\int_{\phi_i}^{\phi}\frac{\sin\phi\cos\phi}{f}\,\mathrm{d}\phi + F_{yi}F_{zi}\left(\frac{1}{E\tilde{A}}-\frac{1}{kG\tilde{A}}\right)\left(\int_{\phi_i}^{\phi}\frac{1}{f}\,\mathrm{d}\phi - 2\int_{\phi_i}^{\phi}\frac{\sin^2\phi}{f}\,\mathrm{d}\phi\right). \tag{20}$$

As for the sectional forces, the y- and the z-components expressed as (F_y, F_z) are constant with respect to s due to the absence of distributed forces. Thus, these components are given by

$$F_y = F_{yi}, \quad F_z = F_{zi}. \tag{21a,b}$$

Equations (17), (19), (20) and (21) are the closed-form solutions of plane elastica considering axial and shear deformations. These closed-form solutions, when applied to a member i, $i+1$, are summarized in Table 2, along with those for the theory of (b) Finite Displacements with Small Strains. In this table, solutions are nondimensionalized, using the member length l and the sectional rigidities $E\tilde{I}$ and $kG\tilde{A}$.

The closed-form solutions for the theory of (b) Finite Displacements with Small Strains can be derived exactly in the same manner as shown in this section. In this derivation, the kinematic relations given by eqns (18) are simplified as follows, using the conditions of small strains:

$$(v_0+y_0)' = \sqrt{g_0}\sin\phi+\Lambda\cos\phi,$$
$$(w_0+z_0)' = \sqrt{g_0}\cos\phi-\Lambda\sin\phi. \tag{22a,b}$$

It should be noted in the above equation that $\sqrt{g_0}$ cannot be approximated by unity, since this approximation yields the inextensional theory.

In Table 2, the following notations are used to express the independent integral components:

$$I_1 = \int_{\phi_i}^{\phi_{i+1}} \frac{1}{f_1}\,\mathrm{d}\phi, \quad I_2 = \int_{\phi_i}^{\phi_{i+1}} \frac{\sin\phi}{f_1}\,\mathrm{d}\phi, \quad I_3 = \int_{\phi_i}^{\phi_{i+1}} \frac{\cos\phi}{f_1}\,\mathrm{d}\phi,$$
$$I_4 = \int_{\phi_i}^{\phi_{i+1}} \frac{\sin\phi\cos\phi}{f_1}\,\mathrm{d}\phi, \quad I_5 = \int_{\phi_i}^{\phi_{i+1}} \frac{\sin^2\phi}{f_1}\,\mathrm{d}\phi \tag{23a–e}$$

where f_1 depends on the theories. As can be seen from Table 2, nondimensionalized solutions are governed by the two independent parameters expressed as

$$\lambda = l\Big/\sqrt{\frac{\tilde{I}}{\tilde{A}}}, \qquad \mu = \frac{E}{kG}, \tag{24a,b}$$

where λ is the slenderness ratio and μ is the ratio of the shear rigidity to the axial rigidity.

If we ignore the shear deformation in Table 2 by setting $\mu = 0$, the solutions of (a) and (b) respectively coincide with those derived from the Bernoulli–Euler beam theory of Finite Displacements with Finite Strains and that of Finite Displacements with Small Strains (Goto *et al.*, 1987.)

In a specific case when μ is equal to unity, f given by eqn (16b) is reduced to

$$f = -\operatorname{sign}(M/E\tilde{I}+\kappa_0)\left\{\left(\frac{M_i}{E\tilde{I}}+\kappa_0\right)^2 - \frac{2F_{yi}}{E\tilde{I}}(\sin\phi-\sin\phi_i) - \frac{2F_{zi}}{E\tilde{I}}(\cos\phi-\cos\phi_i)\right\}^{1/2}. \tag{25}$$

Further, the coefficient $(1/E\tilde{A}-1/kG\tilde{A})$ becomes zero in the governing integral equations of eqns (19) and (20). Thus, excepting the additional terms of $F_{yi}(s-s_i)/kG\tilde{A}$ and $F_{zi}(s-s_i)/E\tilde{A}$ in eqns (19a,b), eqns (17), (19) and (20) for the theory of (a) Finite Displacements with Finite Strains exactly coincide with the corresponding equations for the customary inextensional elastica under the Bernoulli–Euler assumption. As a result, independent integral components are reduced to I_1–I_3 where f_1 is given by

Table 2. Integral solutions

(a) Finite Displacements with Finite Strains	(b) Finite Displacements with Small Strains
$I_1 = 1$	$I_1 = 1$
$\frac{v_{0i+1}+y_{0i+1}}{l} = \frac{v_{0i}+y_{0i}}{l} + \frac{(1-\mu)}{\lambda^2}(B_i I_4 + C_i I_5) + I_2 + C_i \frac{\mu}{\lambda^2}$	$\frac{v_{0i+1}+y_{0i+1}}{l} = \frac{v_{0i}+y_{0i}}{l} + \frac{(1-\mu)}{\lambda^2}(B_i I_4 + C_i I_5) + I_2 + C_i \frac{\mu}{\lambda^2}$
$\frac{w_{0i+1}+z_{0i+1}}{l} = \frac{w_{0i}+z_{0i}}{l} + \frac{(1-\mu)}{\lambda^2}(C_i I_4 - B_i I_5) + I_3 + B_i \frac{1}{\lambda^2}$	$\frac{w_{0i+1}+z_{0i+1}}{l} = \frac{w_{0i}+z_{0i}}{l} + \frac{(1-\mu)}{\lambda^2}(C_i I_4 - B_i I_5) + I_3 + B_i \frac{1}{\lambda^2}$
$A_{i+1} = A_i + C_i I_3 - B_i I_2 + \frac{(1-\mu)}{\lambda^2}\{(C_i^2 - B_i^2) I_4 + C_i B_i (I_1 - 2I_5)\}$	$A_{i+1} = A_i + C_i I_3 - B_i I_2 - \frac{\mu}{\lambda^2}\{(C_i^2 - B_i^2) I_4 + C_i B_i (I_1 - 2I_5)\}$
$B_{i+1} = B_i, \quad C_{i+1} = C_i$	$B_{i+1} = B_i, \quad C_{i+1} = C_i$
$f_1 = -\mathrm{sign}\,(A_i + \kappa_0 l)\Big[(A_i+\kappa_0 l)^2 - 2C_i(\sin\phi - \sin\phi_i) - 2B_i(\cos\phi - \cos\phi_i) - \frac{(1-\mu)}{\lambda^2}\{B_i C_i(\sin 2\phi - \sin 2\phi_i) + \frac{1}{2}(B_i^2 - C_i^2)(\cos 2\phi - \cos 2\phi_i)\}\Big]^{1/2}$	$f_1 = -\mathrm{sign}\,(A_i + \kappa_0 l)\Big[(A_i+\kappa_0 l)^2 - 2C_i(\sin\phi - \sin\phi_i) - 2B_i(\cos\phi - \cos\phi_i) + \frac{\mu}{\lambda^2}\{B_i C_i(\sin 2\phi - \sin 2\phi_i) + \frac{1}{2}(B_i^2 - C_i^2)(\cos 2\phi - \cos 2\phi_i)\}\Big]^{1/2}$

Remarks: $A = \frac{Ml}{E\tilde{I}}, \quad B = \frac{F_z l^2}{E\tilde{I}}, \quad C = \frac{F_y l^2}{E\tilde{I}}, \quad \lambda = l\Big/\sqrt{\frac{\tilde{I}}{\tilde{A}}}, \quad \mu = \frac{E\tilde{A}}{kG\tilde{A}}.$

$$f_1 = -\operatorname{sign}(A_i+\kappa_0 l)\,\{(A_i+\kappa_0 l)^2-2C_i(\sin\phi-\sin\phi_i)-2B_i(\cos\phi-\cos\phi_i)\}^{1/2}. \quad (26)$$

The notations of the above equation are the same as those shown in Table 2.

4. REDUCTION OF ELLIPTIC INTEGRALS TO NORMAL FORMS

The components of the integral equations are given by eqns (23). These integrals are to be transformed to normal forms of elliptic integrals. For simplicity, sign $(A_i+\kappa_0 l)$ included in the function f_1 is assumed unity in the transformation hereinafter. However, the transformation under this assumption will not lose its generality, since sign $(A_i+\kappa_0 l)$ can be let out of the integral sign by dividing the integral interval into subintervals such that sign $(A_i+\kappa_0 l)$ becomes either positive or negative throughout the respective subintervals.

Different from the Bernoulli–Euler beam, f_1 in the Timoshenko beam has the same form in terms of trigonometric functions regardless of the theories and can be expressed as follows:

$$f_1 = (a_0+a_1\cos\phi+a_2\sin\phi+a_3\sin\phi\cos\phi+a_4\cos^2\phi+a_5\sin^2\phi)^{1/2}, \quad (27)$$

where a_0–a_5 are non-zero constants and depend on the theories.

The transformation procedure for the present problem is exactly the same as that shown for the Bernoulli–Euler beam theory of Finite Displacements with Finite Strains (Goto *et al.*, 1987). So, we here make a brief explanation on this procedure.

In the first place, we introduce a new variable $\tilde{x}$ defined by

$$\tilde{x} = \tan\frac{\phi}{2} \quad (-\pi \leqslant \phi \leqslant \pi). \quad (28)$$

Then, the trigonometric integrands of eqns (23) are transformed into algebraic ones as

$$I_i = \int_{\tilde{x}_i}^{\tilde{x}_{i+1}} \frac{R_i}{f_2}\,\mathrm{d}\tilde{x} \quad (i = 1\text{–}5), \quad (29)$$

$$R_1 = 2,\quad R_2 = 4\tilde{x}/(1+\tilde{x}^2),\quad R_3 = 2(1-\tilde{x}^2)/(1+\tilde{x}^2),$$
$$R_4 = 4\tilde{x}(1-\tilde{x}^2)/(1+\tilde{x}^2)^2,\quad R_5 = 8\tilde{x}^2/(1+\tilde{x}^2)^2, \quad (30a\text{–}e)$$

$$f_2 = (b_0\tilde{x}^4+b_1\tilde{x}^3+b_2\tilde{x}^2+b_3\tilde{x}+b_4)^{1/2}. \quad (31)$$

The coefficients of eqn (31) are given by

$$b_0 = a_0-a_1+a_4,\quad b_1 = 2(a_2-a_3),\quad b_2 = 2(a_0-a_4+2a_5),$$
$$b_3 = 2(a_2+a_3),\quad b_4 = a_0+a_1+a_4. \quad (32a\text{–}e)$$

Since f_2^2 is the 4th-order polynominal, it can be known that the integrals given by eqn (29) are expressed by the normal forms of elliptic integrals.

After somewhat cumbersome transformations shown by Goto *et al.* (1987), we can express the integrals of eqns (23) by the linear combination of three normal forms of elliptic integrals along with an elementary integral. Thus, the components of the integrals I_1–I_5 can be symbolically expressed as

$$I_1 \to \tilde{F},\quad I_2 \to \tilde{F}+\tilde{\Pi}+\tilde{G},\quad I_3 \to \tilde{F}+\tilde{\Pi}+\tilde{G},\quad I_4 \to \tilde{F}+\tilde{E}+\tilde{\Pi}+\tilde{G},\quad I_5 \to \tilde{F}+\tilde{E}+\tilde{\Pi}+\tilde{G} \tag{33a–e}$$

where $\tilde{F}$, $\tilde{E}$, and $\tilde{\Pi}$ are the normal forms of elliptic integrals described in Table 3, and $\tilde{G}$ is an elementary integral.

It should be noticed for the Timoshenko beam that the normal forms from the first to the third kind are included both in the solutions for the theory of (a) Finite Displacements with Finite Strains and those of (b) Finite Displacements with Small Strains. This is different from the Bernoulli–Euler beam (Goto *et al.*, 1987).

As explained in Section 3, in the specific case when μ becomes unity, the independent integral components for the theory of (a) Finite Displacements with Finite Strains are reduced to I_1–I_3 with f_1 given by eqn (26). These independent components are exactly the same as those of the inextensional elastica. Thus, making use of the results of Goto *et al.* (1987), I_1–I_3 can be expressed as follows in the manner similar to eqns (33),

$$I_1 \to \tilde{F},\quad I_2 \to \tilde{F}+\tilde{E}+\tilde{G},\quad I_3 \to \tilde{F}+\tilde{E}+\tilde{G}. \tag{34a–c}$$

The above equations do not include the normal form of the third kind, which is replaced here by that of the second kind.

5. NUMERICAL EXAMPLES

Two kinds of simple structures are analyzed to validate the present solutions. In this analysis, the accuracy of the approximate theory of (a) Finite Displacements with Small Strains is examined in comparison with the exact theory of (a) Finite Displacements with Finite Strains. In addition, it is investigated how the shear deformation has an effect on the geometrically nonlinear behavior of beams.

Since the axial force has a significant influence on the geometrical nonlinearity of beams, two kind of structures subjected to axial forces, respectively, compressive and tensile, are chosen as examples. One is a cantilever under an increasing compressive end force together with a small end moment. The other is a beam with hinged ends subjected to an increasing vertical force applied at the midspan. In this structure, a tensile axial force appears with deflection. It should be noted for the beam with hinged ends that the customary inextensional elastica cannot yield accurate solutions, since the assumption of inextensional centroidal axis results in no deflection.

Table 3. Legendre–Jacobi's normal forms of elliptic integrals

The normal elliptic integral of the first kind
$\tilde{F} = \int_{\bar{z}_i}^{\bar{z}_{i+1}} \frac{d\bar{z}}{\sqrt{(1-\bar{z}^2)(1-k^2\bar{z}^2)}}$
The normal elliptic integral of the second kind
$\tilde{E} = \int_{\bar{z}_i}^{\bar{z}_{i+1}} \sqrt{\frac{1-k^2\bar{z}^2}{1-\bar{z}^2}}\, d\bar{z}$
The normal elliptic integral of the third kind
$\tilde{\Pi} = \int_{\bar{z}_i}^{\bar{z}_{i+1}} \frac{d\bar{z}}{(1-\alpha\bar{z}^2)\sqrt{(1-\bar{z}^2)(1-k^2\bar{z}^2)}}$

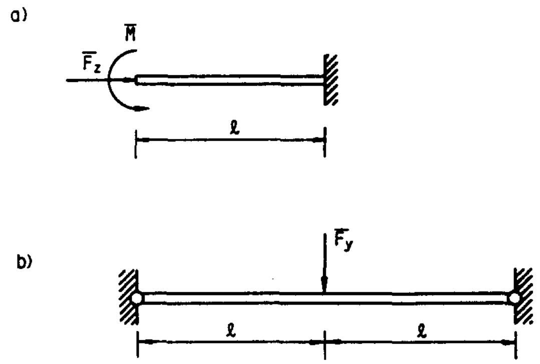

Fig. 2. Examples; (a) cantilever under compressive force and small end moment, (b) beam with hinged ends under vertical load.

These structures are illustrated in Fig. 2, where the member length l is defined for the respective structures.

Solution procedures for the above structures are explained in the following. Since these procedures are exactly the same regardless of the theories, the integral equations for the theory of (a) Finite Displacements with Finite Strains are used for this explanation.

First, a solution procedure is shown for the cantilever. The boundary conditions of this structure are given by

$$F_{yi} = 0, \quad F_{zi} = -\bar{F}_z, \quad M_i = -\bar{M}, \quad v_{0i+1} = 0, \quad w_{0i+1} = 0, \quad \phi_{i+1} = 0. \qquad (35a\text{–}f)$$

Substituting eqns (35a–c,f) into eqn (17) yields the integral equation to calculate the rotational angle ϕ_i of the cross-section. This integral equation cannot be solved explicitly for ϕ_i. Hence, the bisection method is employed here as an iterative method to obtain ϕ_i. Using ϕ_i, so obtained, along with eqns (35), other physical quantities, i.e. v_{0i}, w_{0i}, M_{i+1}, F_{yi+1}, and F_{zi+1} can be directly calculated from eqns (19)–(21) without iteration.

Next, the procedure is explained for the solution of the beam with hinged ends. Making use of the structural symmetry, this problem is reduced to what is shown in Fig. 3. Thus the boundary conditions are expressed by

$$F_{yi} = \bar{F}_y/2, \quad w_i = 0, \quad M_i = 0, \quad v_{0i+1} = 0, \quad w_{0i+1} = 0, \quad \phi_{i+1} = 0. \qquad (36a\text{–}f)$$

Substitution of eqns (36a–c,e,f) into eqns (17) and (19b) gives two integral equations to calculate ϕ_i and F_{zi}. Similar to the case of cantilevers, these integral equations cannot be solved explicitly for ϕ_i and F_{zi}, and the two-variable bisection method is used to calculate these unknown variables. After ϕ_i and F_{zi} are obtained, other physical quantities, i.e. v_{0i}, M_{i+1}, F_{yi+1}, and F_{zi+1}, can be calculated, following the same procedure as is explained for the cantilever.

In the numerical examples, the values of independent structural parameters λ and μ are chosen, based on the following philosophy. Considering that the effect of the axial and the shear deformations becomes more evident in the structures with smaller slenderness ratio, the values of $\lambda = 4$, 5 and 10 are adopted in addition to the usual value of $\lambda = 100$, although such stocky members are not so practical. It should be noted here that $\lambda = 4$ is adopted only for the cantilever, while $\lambda = 5$ is only for the beam with hinged ends. As for

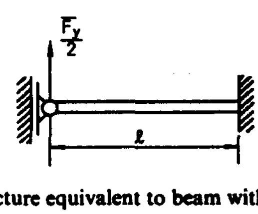

Fig. 3. Structure equivalent to beam with hinged ends.

the ratio of shear rigidity to axial rigidity, the values of $\mu = 0$, 3 and 10 are chosen. The Timoshenko beam with $\lambda = 0$ corresponds to the Bernoulli–Euler beam without shear deformation. $\mu = 3$ is the value for usual beams (Iwakuma and Kuranishi, 1984). $\mu = 10$ represents the value for a beam with smaller shear rigidity.

As a computed result, the load–deflection relationships are summarized in Figs 4 and 5, respectively, for cantilevers and beams with hinged ends, classified according to the values of slenderness ratio. In Fig. 4, the bifurcation loads for the perfect systems are shown for reference. These bifurcation loads can be derived directly from the elliptic integral solutions and this procedure is explained in the Appendix.

We first examine the accuracy of the approximate theory of (b) Finite Displacements with Small Strains in comparison with the exact theory of (a) Finite Displacements with Finite Strains.

In the case when $\lambda = 100$, as seen from Figs 4c and 5c, the load–deflection relationships are represented by one curve, regardless of the theories and the values of μ. Thus, it can be said that the approximate theory under the assumption of small strains has enough accuracy in the analysis of the structures with the usual slenderness ratio of $\lambda = 100$.

However, in case of stocky structures with $\lambda = 4$, 5 and 10, an obvious difference exists between the two theories. For cantilevers, this difference, which becomes most evident around the buckling load, however, decreases in the post buckling range as the horizontal load increases. In this case, the approximate theory of (b) Finite Displacements with Small Strains overestimates the deflection. This tendency is different from that of beams with hinged ends where the approximate theory underestimates the deflection. In the analysis of cantilevers, the discrepancy between the theories is apt to be more pronounced.

In addition to λ, the accuracy of the approximate theory is also influenced by μ, and the smaller value of μ reduces this accuracy. In contrast, as μ takes larger values, the load–displacement curves of the approximate theory approach those of the exact theory. Then, if $\mu = 10$, the approximate theory almost coincides with the exact one even in the case of $\lambda = 4$, 5 and 10.

To summarize the above results, it should be remembered in the application of the theory of (b) Finite Displacements with Small Strains that this approximate theory loses its accuracy specifically for stocky structures with smaller shear deformation.

Next, it is investigated how the nonliner behavior of beams is influenced by the shear deformation. From Figs 4 and 5, as expected, an increase of the value of μ, that is, a decrease of the shear rigidity, results in an increase of the deflections and a decrease of the buckling load for cantilevers. This tendency is more obvious in the structures with smaller slenderness ratio. For cantilevers, the influence of shear deformation, which is most evident around the buckling load, tends to disappear in the post buckling range.

6. CONCLUDING REMARKS

Closed-form solutions with integral expressions were derived for elastica with axial and shear deformations. The theories employed here are the Timoshenko beam theory of (a) Finite Displacements with Finite Strains and that of (b) Finite Displacements with Small Strains, where both the axial and the shear deformations are considered. The solutions, so obtained, include elliptic integrals. Thus, these elliptic integrals were further reduced to Legendre–Jacobi's normal forms in order to utilize the accurate methods for the numerical computation. As a result, it was known that the closed-form solutions for the Timoshenko beam include all the three normal forms, regardless of the theories. However, in the specific case when $\mu = 1$, the solutions for the theory of (a) Finite Displacements with Finite Strains only include the normal forms of the first and the second kinds. The above results are rather different from the extensional elastica without shear deformations.

As numerical examples, two kinds of simple structures under large axial forces, respectively, tensile and compressive, were analyzed. Making use of the calculated results, we examined the effect of the shear deformation together with the accuracy of the approximate theory of (b) Finite Displacements with Small Strains in comparison with the exact theory of (a) Finite Displacements with Finite Strains. As a result, the shear deformation was

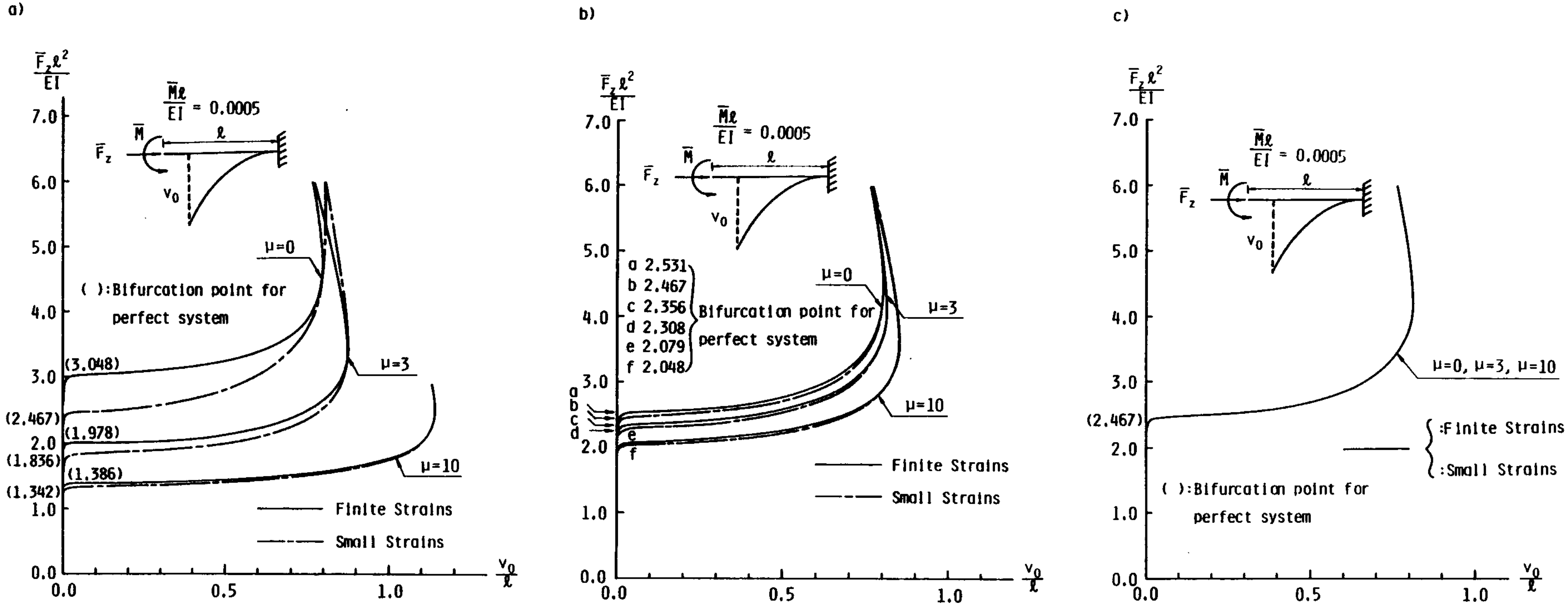

Fig. 4. Load displacement curves of cantilevers; (a) cantilevers of $\lambda = 4$ with three different values of μ, (b) cantilevers of $\lambda = 10$ with three different values of μ, (c) cantilevers of $\lambda = 100$ with three different values of μ.

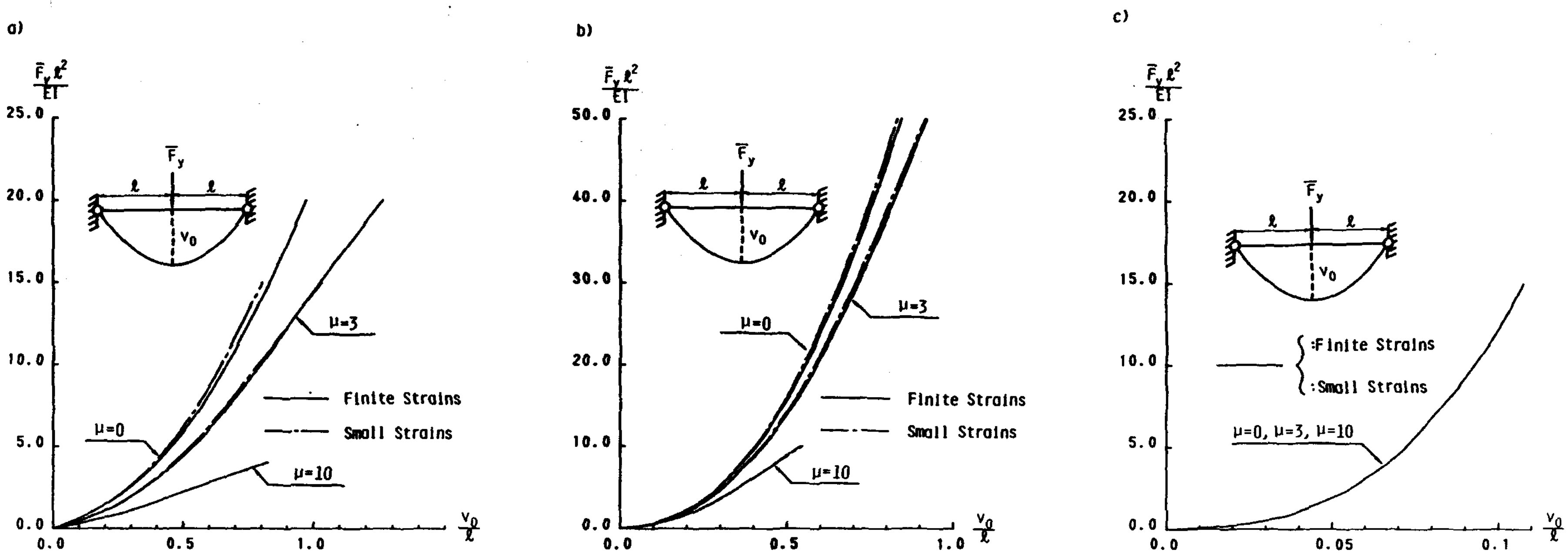

Fig. 5. Load-displacement curves of beams with hinged ends; (a) beams of $\lambda = 5$ with three different values of μ, (4) beams of $\lambda = 10$ with three different values of μ, (c) beams of $\lambda = 100$ with three different values of μ.

found to increase the deflections of the structures and to reduce the buckling loads of cantilevers. This tendency is more pronounced as the slenderness ratio λ becomes smaller. Regarding the accuracy of the approximate theory under the assumption of small strains, the load–displacement curves of this theory deviate from those of the exact theory, as the slenderness ratio λ and the ratio of shear rigidity to axial rigidity μ take smaller values. This means that the approximate theory of (b) Finite Displacements with Small Strains loses its accuracy in stocky beams or columns with less shear deformation.

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APPENDIX: BIFURCATION LOAD OF CANTILEVER

Bifurcation loads are derived from the elliptic integral solutions. The derivation procedure is shown only for the theory of (a) Finite Displacements with Finite Strains because this procedure is the same regardless of the theories.

Since $\bar{M}$ is zero in perfect systems, eqn (16b) is reduced to the following form after the substitution of the boundary conditions given by eqns (36a–c),

$$f = -\left\{\frac{2\bar{F}_z}{\bar{EI}}(\cos\phi - \cos\phi_i) - \frac{\bar{F}_z^2}{2\bar{EI}}\left(\frac{1}{\bar{EA}} - \frac{1}{k\bar{GA}}\right)(\cos 2\phi - \cos 2\phi_i)\right\}^{1/2}. \quad \text{(A1)}$$

Using eqn (A1) along with the transformation defined by

$$\sin\frac{\phi}{2} = \sin\frac{\phi_i}{2}\sin\theta \quad \left(0 \leq \theta \leq \frac{\pi}{2}\right), \quad \text{(A2)}$$

eqn (17) is expressed in terms of θ as

$$l = -\int_0^{\pi/2} \frac{d\theta}{\left[\left(1 - \sin^2\frac{\phi_i}{2}\sin^2\theta\right)\left\{\frac{\bar{F}_z}{\bar{EI}} - \frac{\bar{F}_z^2}{\bar{EI}}\left(\frac{1}{\bar{EA}} - \frac{1}{k\bar{GA}}\right)\left(\cos^2\frac{\phi_i}{2} - \sin^2\frac{\phi_i}{2}\sin^2\theta\right)\right\}\right]^{1/2}}. \quad \text{(A3)}$$

The above equation holds on the bifurcation path and $\bar{F}_z$ approaches the bifurcation load $\bar{F}_{zcr}$ when ϕ_i tends to zero. After the manipulation of $\phi_i \to 0$, eqn (A3) can be integrated as follows:

$$l = -\frac{\pi}{2}\bigg/\sqrt{\frac{\bar{F}_{zcr}}{\bar{EI}} - \frac{\bar{F}_{zcr}^2}{\bar{EI}}\left(\frac{1}{\bar{EA}} - \frac{1}{k\bar{GA}}\right)}. \quad \text{(A4)}$$

Equation (A4) can be solved for $\bar{F}_{zcr}$ as

$$\frac{\bar{F}_{zcr}l^2}{\bar{EI}} = \frac{\pi^2}{2\{1 + \sqrt{1 - \pi^2(1-\mu)/\lambda^2}\}}. \quad \text{(A5)}$$

This buckling load exactly coincides with that derived by Timoshenko and Gere (1961, p. 143) for a helical spring where the change in length of the spring is taken into account during compression. They derived the above formula from a linearized theory.

Following the same procedure, the bifurcation load for the theory of (b) Finite Displacement with Small Strains can be obtained, as shown below, from the corresponding integral equations in Table 2:

$$\frac{F_{xcr}l^2}{EI} = \frac{\pi^2}{2(1+\sqrt{1+\pi^2\mu/\lambda^2})}. \tag{A6}$$

The above also coincides with another formula shown by Timoshenko and Gere (1961, p. 135) for a column with shear deformation. Considering that the axial deformation is small during compression, they ignored it in their derivation.